R^2 2D Quantum Gravity and Dually Weighted Graphs [a]

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A recently introduced model of dually weighted planar graphs is solved in terms of an .elliptic parametrization for some particular collection of planar graphs describing the 2D R^2 quantum gravity. Along with the cosmological constant λ one has a coupling β in the model corresponding to the R^2-coupling constant. It is shown that for any value of β the large scale behavior of the model corresponds to that of the standard pure 2D quantum gravity. On small distances it describes the dynamics of point-like curvature defects introduced into the flat 2D space. The scaling function in the vicinity of almost flat metric is obtained. The major steps of the exact solution are given.

1 Introduction

Matrix models counting the number of planar graphs of various types have played an important role in various domains of physics and mathematics, especially, in the quantitative approaches to two-dimensional quantum gravity and non-critical strings and string field theories.

By means of a standard NxN hermitean 1-matrix with the partition function:

$$Z(t) = \int \mathcal{D}M \; e^{-\frac{N}{2}trM^2 + NtrV(M)}, \tag{1}$$

where the potential is defined by

$$V(M) = \sum_{k=1}^{\infty} \frac{1}{k} t_k M^k, \tag{2}$$

we generate (by expanding $\log Z$ in powers of t_k) all the abstract graphs weighted with the product of factors t_k each corresponding to a vertex with the coordination number k and with the overall factor N^{2-2g} where g is the genus of the graph. The t_k's allow us to control the frequencies of coordination numbers of the vertices, but not the coordination numbers of the faces. One

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of our main technical goals is to generalize the 1-matrix model (1) so that we can also weight the faces with independent weights.

The model (1) has been proven to describe (with the appropriate choices of t_k's) the multicritical points of the pure $2D$ gravity corresponding to the rational conformal matter (of the type (2,2n-1) in the standard classification of the rational 2D conformal theories) [2,3].

An interesting question to ask is whether the 1-matrix model could be used for the description of the pure 2D gravity with higher derivative terms, described by the following formal functional integral:

$$\mathcal{Z}(\lambda,\beta)=\int \mathcal{D}g_{ab}(z)\; e^{-\int d^2z\sqrt{\det g}\;(\mu+\alpha R_g+\frac{1}{\beta_0}R_g^2+\ldots)} \tag{3}$$

On the first sight, from the simple dimensional analysis, apart from the cosmological term (controlled by μ) and the (topological) Einstein term proportional to the curvature R_g, it does not seem meaningful to put further terms into the action of 2D gravity. The simplest term one might want to consider is $\frac{1}{\beta_0}R_g^2$. The bare coupling constant β_0 is however dimensionful and thus should be proportional to the cutoff squared. So it is small and in principle should be dropped, as well as any further higher derivative terms.

On the other hand, if we make β smaller and smaller the characteristic metrics in the functional integral (3) should approach the flat one, since the R^2 suppresses the fluctuations of the metric. So, an interesting question is whether one can get some interesting non-perturbative behavior, where the R^2 coupling could play some essential role. In other words, the question is whether there could be a non-perturbative phase of almost flat metrics, beyond some hypothetic "flattening" phase transition. It is clear that all the known approaches (like the Liouville theory approach) which start from the continuum formulation (3) are not valid for this purpose since the formulation might need serious modifications in this non-perturbative regime. Some nonperturbative, lattice formulation is thus needed.

Unfortunately, the model (1), in spite of the infinite number of independent couplings, cannot be used to study a flattening transition. It is impossible to tune the couplings in such a way that we only generate flat, regular lattices corresponding to a flat metric. The model (1) provides no control over the occurence of different types of faces, so even for only one non-zero coupling, say, $t_k=\delta_{k,4}$, we get a sum over arbitrary ϕ^4 graphs, which describes the highly developed fluctuations of pure 2d quantum gravity.

Hence we have to introduce a model where we can control the numbers of faces with a given coordination number. The most general model of this type, containing the second infinite set of dual couplings t_k^*, which weight the faces

(dual vertices) of graphs, is given by the following partition function (here we fix the genus of the connected graphs $g = 0$):

$$Z(t^*, t) = \sum_G \prod_{v_q^*, v_q \in G} t_q^{*\#v_q^*} \, t_q{}^{\#v_q} \tag{4}$$

where v_q^*, v_q are the vertices with q neighbours on the original and dual graph, respectively, and $\#v_q^*, \#v_q$ are the numbers of such vertices in the given graph G. This expansion is generated by the following matrix model:

$$Z(t^*, t) = \int \mathcal{D}M \; e^{-\frac{N}{2} tr\; M^2 \; + \; Ntr\; V_B(MA)}, \tag{5}$$

with

$$V_B(MA) = \sum_{k=1}^{\infty} \frac{1}{k} \; trB^k \; (MA)^k. \tag{6}$$

A and B are fixed external matrices encoding the coupling constants through

$$t_q^* = \frac{1}{q} \, \frac{1}{N} \; tr \; B^q \qquad \text{and} \qquad t_q = \frac{1}{q} \, \frac{1}{N} \; tr \; A^q. \tag{7}$$

The model generalizes, for $A \neq 1$, the standard one matrix model (1). We will call it the model of dually weighted graphs (DWG).

The DWG model provides the possibility of flattening of the typical graphs. If we take, say, $t_k = t_k^* = \delta_{k,4}$ we will single out the regular square lattice in the partition function. The choice $t_k = \delta_{k,3}, \; t_k^* = \delta_{k,6}$ brings us to the regular hexagonal lattice, dual to the regular triangular lattice. These limits correspond to the $\beta = 0$ limit in the continuum action of (3). Our purpose here is to investigate the critical behaviour of the DWG-model as we approach closer and closer this limit.

2 A Solvable Model of R^2 2D Gravity

In the next section we will show that the DWG-model is in principle solvable for any choice of coupling constants $\{t, t^*\}$. But for our physical purposes we don't need so big space of independent couplings. We have to find the simplest DWG-model with only two independent couplings, one corresponding to the cosmological coupling λ, another to the R^2 coupling β.

We have found that the simplest model of this type corresponds to the following choice of couplings:

$$t_2 = \sqrt{\lambda}\; t, \;\; t_4 = \lambda, \;\; t_6 = \lambda^{\frac{3}{2}} \; \frac{\beta^2}{t}, \; \dots \; t_{2q} = \lambda^{\frac{q}{2}} \; (\frac{\beta^2}{t})^{(q-2)}. \tag{8}$$

With these weights, it is easy to prove, using Euler's theorem, that the partition sum [1] becomes

$$\mathcal{Z}(t,\lambda,\beta) = t^4 \sum_G \lambda^A \ \beta^{2(\#v_2-4)}, \tag{9}$$

where A is the number of plaquets of the graph G and $\#v_2$ the number of positive curvature defects. Note that the latter are balanced by a gas of negative curvature defects, whose individual probabilities are given in (8).

After tuning the bare cosmological constant λ (controlling the number of plaquets) to some critical value $\lambda_c(\beta)$, we expect this model to describe pure gravity in a large interval of β. On the other hand, for λ fixed and $\beta = 0$ we entirely suppress curvature defects except for the four positive defects needed to close the regular lattice into a sphere. It is thus clear that β is the precise lattice analog of the bare curvature coupling β_0 in (3). The phase $\beta = 0$ of "almost flat" lattices – very different from pure gravity – was discussed in detail in [5].

3 Sketch of Solution

3.1 The Itzykson-DiFrancesco formula for the DWG-model

The basic fact which allows to solve the DWG-model is the representation of the matrix integral (5) in terms of the character expansion with respect to the irreducible representations R of the group $GL(N)$, given by Itzykson and DiFrancesco [12]:

$$Z(t,t^*) = c \sum_{\{h^e,h^o\}} \frac{\prod_i (h_i^e-1)!!h_i^o!!}{\prod_{i,j}(h_i^e-h_j^o)} \ \chi_{\{h\}}(A) \ \chi_{\{h\}}(B). \tag{10}$$

The sum runs here over all representations $R\{h\}$ characterized by the Young tableaux with $h_i - i + 1$ boxes in the i-th row. The non-negative integers h_i obey the inequalities:

$$h_{i+1} > h_i. \tag{11}$$

Only the representations with equal number of even(odd) weights $h^e(h^o)$ enter the sum in (10).

The Weyl-Schur characters are defined in the standard way:

$$\chi_{\{h\}}(A) = \det_{(k,l)}\big(P_{h_k+1-l}(\theta)\big) \tag{12}$$

as a determinant of Schur polynomials $P_n(\theta)$ defined through

$$e^{\Sigma_{i=1}^{\infty} z^i \theta_i} = \sum_{n=0}^{\infty} z^n P_n(\theta) \quad \text{with} \quad \theta_i = \frac{1}{i}\, tr[A^i], \tag{13}$$

3.2 The Saddle Point Equation for the Most Probable Representation in the Planar Limit

With the Itzykson-DiFrancesco formula we achieve what is necessary to make the large N limit analytically treatable: the drastic reduction of the number of degrees of freedom. Instead of N^2 integrations over the hermitean matrix M_{ij} we are left with only N summations over the highest weight components $h_1 < h_2 < ... < h_N$. If we assume that the the characteristic $h's$ are of the order N (which will be confirmed by the solution) we can see that the summation factor in (10) is of the order $\exp N$, and as usually one can apply in this situation the saddle point approximation.

For our particular model (8- 9) the saddle point equation (SPE) is obtained by equating to zero the logarithmic derivative of the weight of summation in (10) with respect to h^e or h^o (we assume identical distributions for h^e and h^o). The SPE reads:

$$2F(h) + P\int_0^a dh' \,\frac{\rho(h')}{h-h'} = -\ln h. \tag{14}$$

where

$$F(h_k) = 2\frac{\partial}{\partial h_k^e}\ \ln\ \frac{\chi_{\{\frac{h^e}{2}\}}(a)}{\Delta(h^e)}. \tag{15}$$

and $\rho(h)$ is the density of the highest weight components in the representation R. It is related to the resolvent $H(h)$ of the highest weight components:

$$H(h) = \sum_{k=1}^{N} \frac{1}{h-h_k} = \int_0^a dh' \,\frac{\rho(h')}{h-h'}. \tag{16}$$

Although the details of the derivation of the SPE contain some subtleties, we can roughly say that the term $2F(h)$ comes from the two characters in (10) normalized by the Van-der-Monde determinant $\Delta(h) = \prod_{i>j}(h_i - h_j)$ (one takes $A = B$ for the model dually equivalent to our model), the second term comes from log derivative of $\Delta(h)$ and of the product $\prod_{i,j}(h_i^e - h_j^o)$, and $\log h$ comes from the Sterling asymptotics of $(h^e-1)!!$ (or $h^0!!$).

As has been discussed in detail in [4], this equation actually does not hold on the entire interval $[0, a]$, but only on an interval $[b, a]$ with $0 \le b \le 1 \le a$:

Assuming the equation to hold on $[0, a]$ would violate the implicit constraint $\rho(h) \leq 1$ following from the restriction (11). The density is in fact exactly saturated at its maximum value $\rho(h) = 1$ on the interval $[0, b]$.

A similar saddle point method for the representation expansion was first successfully used in [7] for the calculation of the partition function of the multicolour Young-Mills theory on the 2D sphere (see also [8], [9]).

3.3 Calculation of Characters in the Large N Limit

To solve the SPD (14) for $H(h)$ we have to find a method to calculate effectively the characters, i.e. the function $F(h)$. In this section we describe a method to do it, which is based on the identities for Schur characters, based on the basic properties of Schur polynomials. Leaving aside the details of the derivation (which can be found in our basic paper [1]), we summarize these identities in one equation already in the large N limit. Introducing the function:

$$G(h) = e^{H(h)+F(h)} \tag{17}$$

we have:

$$h - 1 = \sum_{q=1}^{Q} \frac{t_{2q}}{G^q} + \sum_{q=1}^{\infty} \frac{2q}{N} \frac{\partial}{\partial t_{2q}} \ln\Big(\chi_{\{\frac{h^e}{2}\}}(a)\Big)\ G^q, \tag{18}$$

where the coefficients of the positive powers of G in (18) are directly related to the correlators of the matrix model dual to the model defined by the eqs. (8- 9), i.e. the model with potential $V_{A_4}(\tilde{M}A) = \frac{1}{4}\ (\tilde{M}A)^4$:

$$\frac{2q}{N} \frac{\partial}{\partial t_{2q}} \ln\Big(\chi_{\{\frac{h^e}{2}\}}(a)\Big) = \langle \frac{1}{N} tr\ (\tilde{M}A)^{2q} \rangle. \tag{19}$$

We have also assumed for the moment that only a finite number Q of couplings are non-zero (i.e. $t_{2q} = 0$ for $q > Q$). Furthermore, we were able to show in our work [5] that (18) implies the functional equation

$$e^{H(h)} = \frac{(-1)^{(Q-1)}h}{t_Q} \prod_{q=1}^{Q} G_q(h), \tag{20}$$

where the $G_q(h)$ are the first Q branches of the multivalued function $G(h)$ defined through (18) which map the point $h = \infty$ to $G = 0$. The saddlepoint equation (14), together with (20), defines a well-posed Riemann-Hilbert problem. It was solved exactly and in explicit detail in [5] for the case $Q = 2$, where

the Riemann-Hilbert problem is succinctly written in the form

$$\begin{aligned} 2ReF(h) + H(h) &= -\ln(-\tfrac{h}{t_4}) \\ 2F(h) + ReH(h) &= -\ln h, \end{aligned} \tag{21}$$

where $ReH(h)$ denotes the real part of $H(h)$ on the cut $[b, a]$ and $ReF(h)$ denotes the real part of $F(h)$ on a cut $[-\infty, c]$ with $c < b$. This case corresponds to an ensemble of squares being able to meet in groups of four (i.e. flat points) or two (i.e. positive curvature points). We termed the resulting surfaces "almost flat". It turned out that all the introduced functions could be found explicitly in terms of elliptic functions.

The method can be easily generalized to any Q. However, the solution cannot be in general expressed in terms of some known special functions. In the next section we will show that the model of our interest can be solved in terms of elliptic parametrization, more complicated than its particular case, Q=2, of almost flat planar graphs.

Let us also note that the equation (20) can be explicitly solved for $F(h)$ in terms of a contour integral for some wide class of big Young tableaux. Hence we can find the character rather explicitly knowing $H(h)$ and a set of Schur constants $(t_1, t_2, ..., t_Q)$ (see [1]).

3.4 Solution of the Lattice Model of 2D R^2 Gravity

Let us notice that for the model defined by (8, 9) which we chose as a lattice realization of the $2D$ R^2 gravity, the equation (18) contains an infinite number of both negative and positive powers of the G-expansion.

Labeling the first two weights as in [5]: t_2, t_4, we now include all the even t_{2q} with $q \geq 3$, assigning them the following weights: $t_{2q} = t_4 \epsilon^{q-2}$. Equation (18) can then be written compactly as:

$$h - 1 = \frac{t_2}{G} + \frac{t_4}{G\ (G - \epsilon)} + \text{ positive powers of } G. \tag{22}$$

Dropping the positive powers of G (for G small enough) and inverting this equation, we see that there are two sheets connected together by a square root cut running between two finite cut points, d and c. On the physical sheet a further cut (running from b to a), corresponding to $e^{H(h)}$, connects to further sheets.

The equations (21) now read:

$$\begin{aligned} 2ReF(h) + H(h) &= -\ln(\tfrac{h}{\epsilon t_2 - t_4}) \\ 2F(h) + ReH(h) &= -\ln h, \end{aligned} \tag{23}$$

the first coming from the large N limit of the character (20) and the second, in view of (16), being the saddlepoint equation (14). These two equations tell us about the behaviour of the function $2F(h)+H(h)$ on the cuts of $F(h)$ and $H(h)$, respectively. We have introduced the notation $ReF(h)$ to denote the real part on the cut of $F(h)$, and similarly for $ReH(h)$. The principal part integral in (14) is thus denoted in (23) by $ReH(h)$.

Our object is to now reconstruct the analytic function $2F(h)+H(h) = 2\ln G(h) - H(h)$ from its behaviour on the cuts. To do this we first need to understand the complete structure of cuts. We already know the structure of cuts of $H(h)$; it has a logarithmic cut running from $h=0$ to $h=b$, corresponding to the portion of the density which is saturated at its maximal value of one, and a cut from b to a corresponding to the "excited" part of the density, where the density is less than one. It thus remains for us to understand the cut structure of $\ln G(h)$.

The function $G(h)$ has two cuts on the physical sheet. The first cut, running from b to a, corresponds to the cut of $e^{H(h)}$, the second cut, running from cut point c to cut point d, corresponds to the cut of $e^{F(h)}$. To see whether $\ln G(h)$ has any logarithmic cuts, we first notice from (22) that $G(h)$ is non zero everywhere in the complex h plane except possibly at infinity. Thus for $\ln G(h)$ the only finite logarithmic cut points are at $h=b$, defined to be the end of the flat part of the density (this corresponds to the end of the cut of $e^{H(h)}$), and possibly the cut point c, defined to be at the end of the cut of $e^{F(h)}$. The only remaining question is whether this logarithmic cut starting at $h=b$ goes off to infinity or terminates at c. For large h we see from (22) that $\ln G(h) = \ln\epsilon + \mathcal{O}(\frac{1}{h})$, i.e. there is no logarithmic cut at infinity. We conclude that the cut structure of the function $\ln G(h)$ consists of the cuts corresponding to $e^F(h)$ and $e^{H(h)}$ connected together by a logarithmic cut, whose cut points are b and c.

We thus understand the behaviour of $2F(h)+H(h)$ on all of its cuts. Standard methods then allow us to generate the full analytic function $2F(h)+H(h)$. The four cut points, a and b defining the cut of $H(h)$, and c and d defining the cut of $F(h)$ are fixed later by boundary conditions.

Without going into the details of the solution let us present at least one final result: the density of weights $\rho(h)$ of the most probable representation $R\{h\}$. It can be obtained from $2F(h)+H(h)$ by using the saddle point equation $2F(h)+ReH(h) = -\ln h$ and the fact that the resolvent (16) for the Young tableau can be written as $H(h) = ReH(h) \mp i\pi\rho(h)$. To obtain the result one has to use all the analytical information listed above. After some tedious

calculations we obtain:

$$\rho(h) = \frac{u}{K} - \frac{i}{\pi} \ \ln\left[\frac{\theta_4\big(\frac{\pi}{2K}(u-iv),q\big)}{\theta_4\big(\frac{\pi}{2K}(u+iv),q\big)}\right], \tag{24}$$

where v and u are defined by

$$v = sn^{-1}\big(\sqrt{\frac{a-c}{a-d}},k'\big), \qquad u = sn^{-1}\big(\sqrt{\frac{(a-h)(b-d)}{(a-b)(h-d)}},k\big), \tag{25}$$

and

$$k^2 = \frac{(a-b)(c-d)}{(a-c)(b-d)}. \tag{26}$$

K and K' are the complete elliptic integrals of the first kind with respective moduli k and $k' = \sqrt{1-k^2}$. E is the complete elliptic integral of the second kind with modulus k, $E(v,k')$ is the incomplete elliptic integral of the second kind with argument v and modulus k' and sn, cn and dn are the Jacobi Elliptic functions. The nome q is defined by

$$q = e^{-\pi\frac{K'}{K}}. \tag{27}$$

To fix the constants a, b, c and d, we expand $i\pi\rho(h) = 2F(h)+H(h)+\ln h$ for large h and compare the resulting power series expansion to that obtained from inverting (18):

$$2F(h)+H(h) = 2\ln\epsilon + \big(\frac{2t_4}{\epsilon^2}-1\big)\frac{1}{h} \quad + \quad \mathcal{O}\big(\frac{1}{h^2}\big). \tag{28}$$

The terms of $\mathcal{O}\big(\frac{1}{h^2}\big)$ depend on the as yet unknown positive powers of G in (22).

Afinal boundary condition is fixed by the normalization of the density.

Using this information we can calculate the physical quantities of interest, for example, the derivative of the free energy with respect to the cosmological constant. Denoting the free energy by $\mathcal{Z}(t,\lambda,\beta)$, $Z = e^{-N^2\mathcal{Z}}$, we have

$$\frac{\partial}{\partial\lambda}\mathcal{Z}(t,\lambda,\beta) = \frac{1}{4\lambda}(\langle\frac{1}{N}trM^2\rangle - 1). = \frac{1}{2} + \langle h\rangle \tag{29}$$

where $\langle h\rangle = \int dh \ \rho(h) \ h$. The result of the last integration, as well as the conditions for a,b,c,d are too bulky to be presented here (see [1] for the details). In the next section we will present already the physical consequences of it, namely, the universal expression for the free energy near the flattening limit.

4 Physical Results and Conclusions

One needs to do quite tedious calculations to extract the physical results from our solution summerized in the equations (24- 29). The details can be found in [1] and here we present only the essential results:

1. We found the equation for the critical curve of pure gravity in the parameter space (λ, β). It did not show any "flattening" phase transition in the whole interval $0 < \beta < \infty$. Pure gravity appears to be the only possible critical behaviour for the large scale fluctuations of the metric (the only infrared stable critical point) within our model. This should be true for a big class of similar lattice models of pure R^2 gravity in 2 dimensions. For example, if we would allow only the deficits of angle being $0, \pm\pi$ in the vertices of our quadrangulation, this physical conclusion should not change. On the other hand, the multicritical behaviours can occur for a more complex parameter space (more t^*, t couplings to vary) and we could expect some new phase structures. Sums over graphs with the sign-changing Boltzmann weights might well be necessary for this.

2. Once we realize that nothing interesting happens in our model for finite values of β we should concentrate our attention on the double limit $\beta \to 0, \lambda \to \lambda_c(\beta) = 1 - \frac{\pi}{\sqrt{2}}\beta + O(\beta^2)$ with the double limit parameter $x = 1 + \frac{\sqrt{2}}{\pi\beta}(\lambda_c - \lambda)$ fixed. Note that this is a very natural, dimensionless scaling parameter; $\lambda_c - \lambda$ is the continuum cosmological constant with dimension of inverse area and β controls the number of curvature defects per unit area (and thus also has dimension of inverse area).

We arrive at the free energy in the vicinity of the (locally) flat metric:

$$\mathcal{Z}(t, \lambda, \beta) = \frac{4t^4}{15\beta^2}\left[x^6 - \frac{5}{2}x^4 + \frac{15}{8}x^2 - \frac{5}{16} - x(x^2-1)^{5/2}\right]. \qquad (30)$$

This scaling function exhibits two different behaviours in the two opposite limits:

1. $x_c \to 1$ - the pure 2D gravity regime for any finite positive β and $\lambda \to \lambda_c(\beta)$:

$$\mathcal{Z}(t, \lambda, \beta) \sim (\lambda_c - \lambda)^{5/2} \qquad (31)$$

This corresponds to the standard value of the string susceptibility exponent $\gamma_{str} = -1/2$.

This results mean that even for very big but finite R^2-coupling β^{-1} we will always find on big enough distances the metric fluctuations obeying the pure gravity scaling. In some sense one can say, that the R^2 2D gravity does not exist as a special phase of the 2D gravity. The R^2 term can be always

dropped from the action in the long wave limit. Our results demonstrate this nonperturbatively, starting from a solvable lattice model.

2. In the opposite limit $x \to \infty$, i.e., $\beta \to 0$ with λ fixed, we obtain:

$$\mathcal{Z}(t, \lambda, \beta) = \frac{\pi^2}{48} \frac{\beta^4}{(1-\lambda)^2} \tag{32}$$

This limit corresponds to a sum over metrics flat everywhere except of a finite number of curvature defects. One can easily see from the purely combinatorial calculations that (32) corresponds to the sum over all flat metrics with only four defects with the deficit of angle π. Expanding (30) as a power series in β (the first term of which is given by (31) we see that the second term in the series expansion corresponds to the the sum over all flat metrics with five π defects plus one $-\pi$ defect, the third to six π defects plus two $-\pi$ defects, etc.

This limit does not appear to describe a new phase of smooth metrics. It corresponds instead to the statistics of a dilute gas of curvature defects introduced in an otherwise flat metric. It has nothing to do with the 2D gravity but it is an interesting statistical mechanical model in itself. More than that, one can prove that the dominating graphs are not lattice artifacts: in this limit the manifold "forgets" that it is built out of squares. It consists of big flat patches with sparsely distributed point-like curvature defects introduced ono it in such a way that they close the surface into a manifold of spherical topology. If one considers this object as a sphere punctured at the points where curvature defects occur, one can show by the direct calculations[c], for the simplest configurations, that the corresponding moduli parameters do not fall on the boundaries of the moduli space but are typically in some general position inside the fundamental domain.

It would be nice to solve this model (and may be more general cases including the matter fields) by a continuum approach. In the Liouville theory picture the curvature defects correspond to coulomb charges floating in a two-dimensional parametrization space. This could be an alternative approach to 2D quantum gravity in general.

Let us conclude by noting that our analytical approach to the models of dually weighted graphs can be generalized to a big class of other matrix models which could be of physical, as well as of the mathematical interest. For example, one can formulate a generalized two-matrix model with partition function:

[c] we thank Paul Zinn-Justin for clarifying this question for us

$$Z = \int d^{N^2}X d^{N^2}Y \exp Ntr(F(XY) + U(X) + V(Y)) \qquad (33)$$

where X, Y are the NxN hermitean matrices and F, U, V are some arbitrary functions. The character expansion allows us again to reduce the number of integration (or summation) variables to $\sim N$ thus making it possible to consider it by means of the saddle point methods. Another, more general solvable k-matrix model of this type is:

$$Z = \int (\prod_{j=1}^{k} d^{N^2}X_j) \exp Ntr\big(F(\prod_{j=1}^{k} X_j) + \sum_{j=1}^{k} U_j(X_j)\big) \qquad (34)$$

The $\exp NtrF(\prod_{j=1}^{k} X_j)$ factor can be also expended in characters and one can then integrate over the angular degrees of freedom of the matrices by the use of only the simple SU(N)-orthogonality relations. This approach is at the heart of the Itzykson-DiFrancesko formula which we extensively used for our present model of lattice 2D R^2 gravity.

A big question left is how to generalize our approach to include of the matter fields on our lattice manifold. The corresponding matrix models containing the R^2 type couplings can be easily formulated but cannot be solved by the character expansion methods presented here. This is not surprising, since models of interacting spins on the 2D regular lattice, even the integrable ones, are very complicated. Nevertheless, we think that our methods could eventually be powerful enough to advance in this direction and provide a missing link between two branches of mathematical physics: integrable statistical mechanical models on regular 2D lattices, on the one hand, and on random dynamical lattices, on the other hand. In more physical language, it could provide a link between the integrable models of interacting fields with and without the 2D quantum gravity fluctuations.

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